APPLICATION OF ROBUST ESTIMATION METHODS FOR THE ANALYSIS OF OUTLIER MEASUREMENTS

APLIKÁCIA ROBUSTNÝCH ODHADOVACÍCH METÓD PRI ANALÝZE ODĽAHLÝCH MERANÍ

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Abstract

The basis of mathematical analysis of geodetic measurements is the method of least squares (LSM), whose bicentenary we celebrated in 2006. In geodetic practice, we quite often encounter the phenomenon when outlier measurements penetrate into the set of measured data as a result of e.g. the impact of physical environment. That fact led to modifications of LSM that have been increasingly published mainly in foreign literature in recent years. The mentioned alternative estimation methods are e.g. robust estimation methods and methods in linear programming. The aim of the present paper is to compare LSM with the robust estimation methods on an example of a regression line.

Abstrakt

Základom matematickej analýzy dát je metóda najmenších štvorcov (MNŠ), ktorej dvesté výročie sme si pripomenuli v roku 2006. V geodetickej praxi sa v dôsledku napr. vplyvu fyzikálneho prostredia pomerne často stretávame s javom, že do súboru meraných dát prenikajú odľahlé merania. Práve táto skutočnosť viedla k modifikáciám MNŠ ktoré sa v posledných rokoch čoraz častejšie publikujú predovšetkým v zahraničnej literatúre. Spomínanými alternatívnymi odhadovacími metódami, sú napr. robustné odhadovacie metódy a metódy lineárneho programovania. Náplňou predloženého príspevku je porovnanie MNŠ s robustnými odhadovacími metódami na príklade regresnej priamky.

Key words: robust estimations, method of least squares, outlier detection

1 INTRODUCTION

As declared above the subject of the present contribution is to refer also to other ways of processing the results of geodetic measurements as compared to the standard method being used, which is the least squares method (LSM). Although this method is completely remade, when using it a "smudge" of undetected blunders penetrated into the set of measured values may occur, mostly as a result of the undue impact of physical environment on measurements. For this reason, the above mentioned alternative methods of statistical processing appeared in geodetic practice. Given the fact that the alternative estimation methods are used primarily for the detection of troublesome effects on measurements, the properties of these processing methods are applied to an illustrative example of a regression line weighted first by a normal distribution and then to an example of a line weighted by an experimental outlier. In this paper the following methods are compared with LSM: the robust M-estimator by Huber, the robust M-estimator by Hampel and the Danish method.

2 THE METHOD OF LEAST SQUARES

The need of an adjustment calculus and the discovery of LSM itself resulted from advances in technology for measuring and accumulation of surveying material in the field of astronomy and geodesy at the end of the 17th century. The discoverer of the least squares method, having become a classic tool for the theory of errors, is *Carl F. GAUSS (1777-1855)* [7]. The essence of this method consists in minimizing the sum of squared deviations occurred during the measurement of the behaviour of a quantity or physical phenomenon (3). The least squares method results from the condition of a so-called *L2-norm* (2), whereas the norm is the number assigned to each n-dimensional vector $v = (v_1, v_{2,\dots,} v_n)$ characterizing its size in some sense [1] , [2]. In geodesy, the objective functions of the following type are used most frequently:

$$\rho(v) = \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} = \min. \qquad i \in \langle 1, n \rangle \tag{1}$$

where:
p - parameter defining the special type of objective function,
in $_i$ - vector of corrections.

Assuming $p = 2$ *(L2* norm) the objective function is as follows:

$$\rho(v\ ) = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}} \tag{2}$$

leading to the least squares method that, under certain conditions, leads to the most reliable estimators of unknown quantities, and hence this is the method most commonly used in geodetic practice for processing measured data. The mathematical formulation of this method is as follows:

$$\sum p_i v_i^2 \equiv \mathbf{v}^T \mathbf{P} \mathbf{v} = \min.$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} p_1 & 0 & . & 0 \\ 0 & p_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & 0 & p_n \end{bmatrix} \quad k \text{ const., for } i=1,2, \dots n \tag{3}$$

where:
v - vector of corrections
P - matrix of weighting coefficients (weights of p measurements) being ordered along the diagonal of the weighting matrix.

The measurement weights are proportional numbers, qualitatively evaluating the achieved measurement result. Introducing the weights we prefer a more accurate measurement that takes a share in the measurand adjustment.

The least squares method will also be explained on a one-dimensional linear model, while all the estimation methods will be demonstrated on an example of a regression line. Let the following linear relationship exists between the variable *y*, variables *X* (Fig. 1) and the random component u_i:

$$\begin{aligned} &y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + u_i \quad \text{kde } i = 1,2\cdots n \\ &\mathrm{y} = \beta_0 + \beta X + u \quad \text{linear model equation} \end{aligned} \tag{4}$$

where:

y_i - dependent variables

$X_1, X_2, \ldots X_k$ - independent variables

β_0 - model parameter expressing the value attained by y when X equals zero, called *intercept*

β - *regression coefficient* indicating the slope of the regression line, i.e. the rate at which y_i increases per unit increase in $X_1, X_2, \ldots X_k$

u_i – random component.

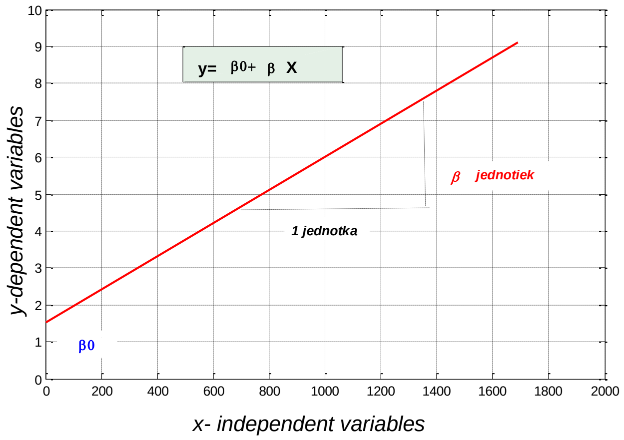

Fig. 1 One-dimensional linear model

We may rewrite the model using the following matrix notation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}\,, \tag{5}$$

where:

y - *n*-dimensional vector of the dependent variables,

X - matrix (k +1) independent variables

u - n-dimensional vector of the random components,

$\boldsymbol{\beta}$ - unknown (unobservable) parameters we need to determine; in geodesy we use the term estimators.

Thus, when deriving an estimator we result from the matrix notation (5). The problem lies in finding the estimator $\hat{\beta}$ so that the estimated regression line $\hat{\mathbf{y}} = \mathbf{X}\hat{\beta} + \mathbf{u}$ approximates best the regression line $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$.

The following applies to the difference $\mathbf{y} - \hat{\mathbf{y}}$, the so-called vector of residuals (the vector of corrections in geodesy) $\mathbf{v} = \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$:

$$\mathbf{v} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}} \text{ z čoho}$$
$$\mathbf{y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{v} \tag{6}$$

whereof

Using LSM, we search for the estimator $\hat{\boldsymbol{\beta}}$ of the vector of parameters β so that the sum of the squared residuals (3) is minimum.

$$\sum_{i=1}^{n} v_i^2 = \mathbf{v}^T\mathbf{v} = (\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})^T(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \tag{7}$$

Multiplying (7) we get:

$$\begin{aligned} \mathbf{v}^T v &= \mathbf{y}^T\mathbf{y} - 2\mathbf{X}^T\hat{\boldsymbol{\beta}}^T\mathbf{y} + \mathbf{X}^T\boldsymbol{\beta}^{\mathbf{T}}\mathbf{y} - \mathbf{y}^T\mathbf{X}\hat{\boldsymbol{\beta}} + 2\mathbf{X}^T\hat{\boldsymbol{\beta}}^T\mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{X}^T\hat{\boldsymbol{\beta}}^T\mathbf{X}\hat{\boldsymbol{\beta}} \\ &= \mathbf{y}^T\mathbf{y} - \mathbf{X}^T\hat{\boldsymbol{\beta}}^T\mathbf{y} - \mathbf{y}^T\mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{X}^T\hat{\boldsymbol{\beta}}^T\mathbf{X}\hat{\boldsymbol{\beta}}. \end{aligned} \tag{8}$$

Let us differentiate the equation (8) with respect to $\hat{\boldsymbol{\beta}}^T$:

$$\frac{\partial \mathbf{v}^T\mathbf{v}}{\partial \hat{\boldsymbol{\beta}}^T} = -\mathbf{X}^T\mathbf{y} + \mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \tag{9}$$

If we set the derivative (9) equal to zero, then we obtain the relationship

$$\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{y} \tag{10}$$

that is the matrix notation of normal equations. The matrix $\boldsymbol{X}^T\boldsymbol{X}$ is nonsingular, i.e. it is invertible. Therefore, the following applies to the vector of unknown parameters from the relation (10):

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} \tag{11}$$

If the line equation (Fig. 2) has the generally known form

$$y = aX + b \tag{12}$$

where *a and b* are the searched estimators to be determined, the line equation has the following form in a matrix notation:

$$\mathbf{y} = \mathbf{A}X + \mathbf{v}\,, \tag{13}$$

where:

v: vector of corrections,

A: configuration matrix, the matrix of partial derivatives,

X: searched estimators (in geodetic calculations, X is replaced by the parameter Θ, these are coordinates as a rule).

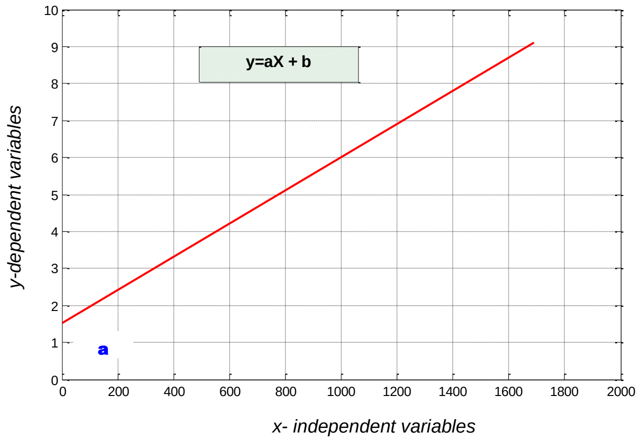

Fig. 2 Regression line in general

Whereas the relationships between the measured and the unknown variables are expressed by an intermediary function of the searched unknown parameters (estimators), the given model may be rewritten into the following form:

$$\begin{aligned} \mathbf{y} &= \mathbf{l} + \mathbf{v} \\ \mathbf{l} + \mathbf{v} &= \mathbf{A}\hat{\Theta} \end{aligned} \tag{14}$$

configuration matrix

$$\mathbf{A} = \frac{\partial y}{\partial a}, \frac{\partial y}{\partial b} \tag{15}$$

$$\begin{aligned} \mathbf{l} &= \mathbf{y} \\ \mathbf{l} + \mathbf{v} &= \mathbf{A}\hat{\Theta} \end{aligned}$$
$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} + \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \tag{16}$$

The following applies to the vector of residuals:

$$\mathbf{v} = \mathbf{A}\hat{\Theta} - \mathbf{l}\,, \tag{17}$$

which leads to the Guss-Markov model:

$$\begin{aligned} \mathbf{v} &= \mathbf{A}\hat{\Theta} - \mathbf{l} \\ \textstyle\sum_l &= \sigma_0^2 \mathbf{Q}_l \end{aligned} \tag{18}$$

When using LSM the following applies to the sought estimators after adjusting the parameters (indices):

$$\hat{\mathbf{\Theta}} = (\mathbf{A}^T \mathbf{Q}_l^{-1} \mathbf{A})^{-1} \left(\mathbf{A}^T \mathbf{Q}_l^{-1} \mathbf{l}\right),$$

$$\text{matrix } \mathbf{Q}_l = \mathbf{I} \text{ (unit)} \qquad . \tag{19}$$

$$\hat{\mathbf{\Theta}} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} (\mathbf{A}^T \mathbf{l}) \cdots \text{analogously } (11)$$

3 ROBUST ESTIMATION METHODS

In the second half of the last century unconventional estimation methods were developed in the theory of linear programming in addition to the standard estimation methods. In those methods, other variable, not the arithmetic mean, is selected as the parameter of centrality. From these methods, the so-called *robust estimation methods* are preferred in geodetic practice [10] , [9] , [8] , [6] , [5] , [4].

Two types of robust estimations are known: *robust estimations applied on the basis of LSM*, when the sum of squared corrections is replaced by more appropriate correction functions: e.g. maximum credible estimators (robust M - estimators) [10] , [9] , [8] , [6] , [5] , [4] and *pure robust methods that include* linear programming methods, such as a s*implex method.*

The M robust method of adjustment on the principle of LSM occurs when there is no minimized function $v^T v$ in the estimation process, but another suitably chosen correction function $\rho(v_i)$ called the *function of losses (estimators).*

$$\rho(v_i) = \min \quad , \tag{20}$$

that generates the so-called *influence function* $\Psi(v_i)$ for the estimation process, characterizing the impact of errors on the adjusted values, which the following applies to:

$$\sum \Psi(v_i) = 0 \, , \tag{21}$$

where:

$$\Psi(v_i) = \frac{\partial \rho(v_i)}{\partial v_i} \quad . \tag{22}$$

In order to be robust, the adjustment should be performed by an iterative method with variable weights, i.e., so that the weights for observables are determined as functions of corrections in each iteration step

$$p(v_i) = \frac{\Psi(v)}{v} \, , \tag{23}$$

where $p(v_i)$ is the weighting function in the adjustment process solution. The iterative robust estimation algorithm proceeds as follows:

1. In the first iteration step the standard LSM adjustment with weights $p_1^{(1)} = 1$ is carried out ((1)- the iterative step for the i-th observable), in case of heterogeneous measurements it is necessary to perform their homogenization by means of the matrix of the square roots of the weights $\sqrt{\mathbf{P}}$ (this feature is available in a MATLAB program - rootm (P)) [4] .
2. From the corrections obtained in the first iteration step using the weighting function $p(v_i)$ the new weights are determined to be used in the next step and by analogy in further ones.

New weights are created according to a relevant regulation for the weighting function up to the end of the iteration process being selected with a reasonable number of steps so that the acceptable convergence of weights occurs in the last steps. The weighting functions are determined based on previously theoretically and empirically investigated and verified different assumptions and for different kinds of measurements.

Using the robust estimation techniques in the processing of horizontal geodetic control networks a great deal of attention was paid to v [4]; for that reason the present contribution demonstrates the following robust estimation methods with their experimented and verified constants on an example of a regression line [3] , [4] , [5] , [6] , [8] , [9] , [10] : ***the robust M - estimator by Huber, the robust M - estimator by Hampel and the Danish method.***

2.1 Robust M-estimator by Huber

This estimator uses the following functions and relevant *tuning*- *damping* constants

estimation function

$$\rho(v) = \begin{Bmatrix} \frac{1}{2}v^2 & |v| < k \\ k.|v| - \frac{1}{2}k^2 & |v| \geq k \end{Bmatrix} \quad \text{The value of the damping constant is usually } 1{,}5 \tag{24}$$

influence function

$$\Psi(v) = \frac{\partial\rho(v)}{\partial(v)} = \begin{Bmatrix} v & |v| < k \\ k.sign(v) & |v| \geq k \end{Bmatrix} \tag{25}$$

weighting function

$$p(v) = \frac{\Psi(v)}{|v|} = \begin{Bmatrix} 1 & |v| < k \\ \frac{k}{|v|} & |v| \leq k \end{Bmatrix} \tag{26}$$

The behaviours of these individual functions may be illustrated graphically as well [4] , [10]. Due to the scope of the contribution and the fact that the principle of robust M-estimation methods consists in the repetition of LSM with gradual changing the weights according to the relevant regulation (23) at the used processing methods, we present only the graph of the weighting function.

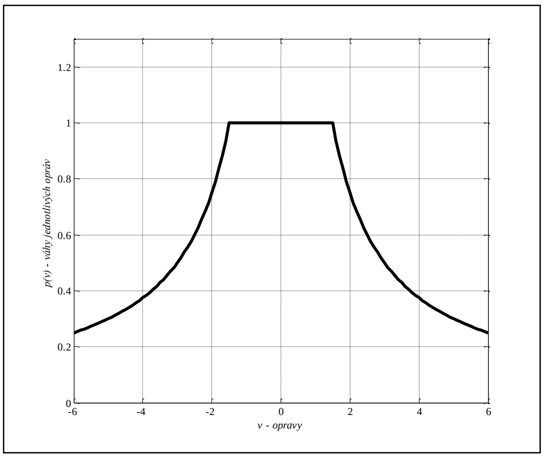

Fig. 3 Graphic behaviour of the weighting function for the robust M-estimator by Huber

2.2 Robust M-estimator by Hampel

This estimator uses the following functions and relevant damping constants:

estimation function

$$\rho(v) = \left\{ \begin{array}{ll} \frac{1}{2}v^2 & |v| < a \\ a.|v| - \frac{1}{2}a^2 & a \le |v| < b \\ ab - \frac{1}{2}a^2 + & \\ \frac{1}{2}a(c-b)\left[1 - \left(\frac{c-|v|}{c-b}\right)^2\right] & b \le |v| < c \\ ab - \frac{1}{2}a^2 + \frac{1}{2}(c-b)a & c \le |v| \end{array} \right\} \tag{27}$$

influence function

$$\Psi(v) = \left\{ \begin{array}{ll} v & |v| < a \\ a.sign(v) & a \le |v| < b \\ a.\frac{c-|v|}{c-b}.sign(v) & b \le |v| < c \\ 0 & c \le |v| \end{array} \right\} \tag{28}$$

weighting function

$$p(v)=\begin{cases}1 & |v|<a\\ \dfrac{a}{|v|} & a\le|v|<b\\ a.\dfrac{c-|v|}{(c-b)|v|} & b\le|v|<c\\ 0 & c\le|v|\end{cases} \tag{29}$$

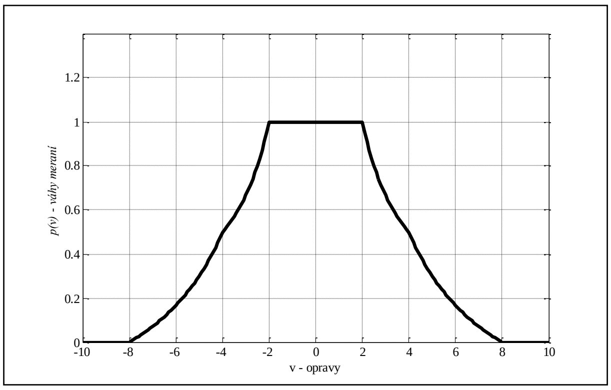

Fig. 4 Graphic behaviour of the weighting function for the robust M-estimator by Hampel

2.3 Danish method

The strategy of the Danish method is that it reduces (shrinks) the impact of remote measurements (outliers) on the estimates of quantities. The principle of this method is based on the indication of the outliers by corresponding major corrections. After the standard adjustment of the estimations of the first-order parameters by the method of least squares through the Gauss - Markov model, the a priori weights of measurements are replaced by correction functions. The next iteration step is the adjustment of the initial weights of measurements according to the following relation

$$p_{i+1}=p_i\,p(v) \qquad i=1,2,\cdots \tag{30}$$

which results in the increase of the absolute values of outlier measurements, while reducing their deformation impact on the network geometry. The iterative cycle is repeated until expected results are achieved. In the current horizontal geodetic control networks the solution requires no more than (10-15) iterations [4] , [3]. For this method the following functions and standards have been derived:

influence function:

$$\Psi(v)=\begin{cases}|v| & |v|<c\\ v.e^{-a|v|^b} & |v|\ge c\end{cases} \tag{31}$$

At present different types of exponential functions are used for the weighting function [3]. In order to demonstrate this method, the following weighting function was used in the practical solution:

<u>*weighting function*</u>

$$p(v_i) = \begin{cases} 1 & \text{pre} \quad \dfrac{p_i |v_i|}{s_0} < c \\ \exp\left(-\dfrac{|v_i p_i|}{c s_0} \right) & \text{pre} \quad \dfrac{p_i |v_i|}{s_0} > c \end{cases} \tag{32}$$

with these standards:

for the 1st iteration step $a = 0$,

for the 2nd and 3rd iteration steps $a = 0.05$ and throughout the iterative process $b = 3, c = 3$.

Thus, the equation (32) is defined by the following interval:

$$-c < |v_i| \sqrt{p_1} / s_0 < c \tag{33}$$

which the weights of measurements are started to be determined from. The constant *c* is chosen usually between 2 to 3 and depends on the redundancy (excess measurements) determined by the Gauss - Markov model and the quality of measurands. If the constant c <2 the method being used is robust, in case c> 3 the processing method is changed to LSM.

4 EMPIRICAL DEMONSTRATION

This part of the paper is devoted to examining the properties of individual estimation methods using an example of a regression line that may be represented from the geodetic point of view, e.g. by the electronic rangefinder equation.

4.1 Adjustment of the regression line through LSM

The properties of estimation methods were studied on an illustrative example of a regression line $y = ax + b$ whose modified form $\sigma_d = b + a.d.ppm$ is normally used in geodesy to characterize the accuracy of electronic rangefinders, where the parameter a characterizes the effect of independent errors and the parameter b characterizes the effect of errors dependent on measured distance. The constant ppm (parts per million - means "out of million") is equal to 10^{-6}. The adjusted regression line in the form $y = 2ppmd + 3[mm]$ was weighted with a normal distribution by means of the least squares method (Fig. 5).

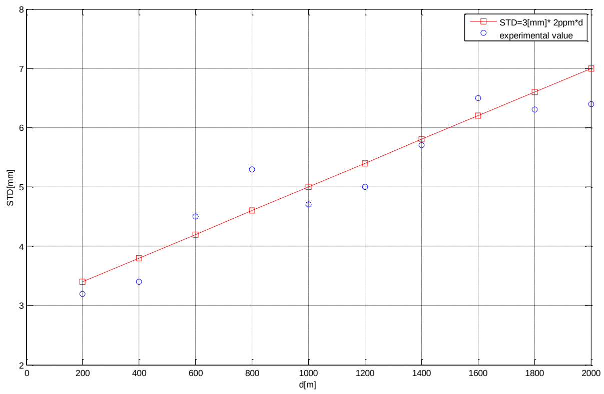

Fig. 5 Deterministic model of the regression line weighted with a normal distribution

Note: $STD = \sigma_d$,

When using this method the basis is the model $\mathbf{l} + \mathbf{v} = \mathbf{A}\hat{\mathbf{\Theta}}$ leading to the famous Gauss-Markov model [1]. The results of adjustment of the regression line by means of LSM are demonstrated in Table 1, the graphical interpretation of the results obtained is shown in Fig. 6.

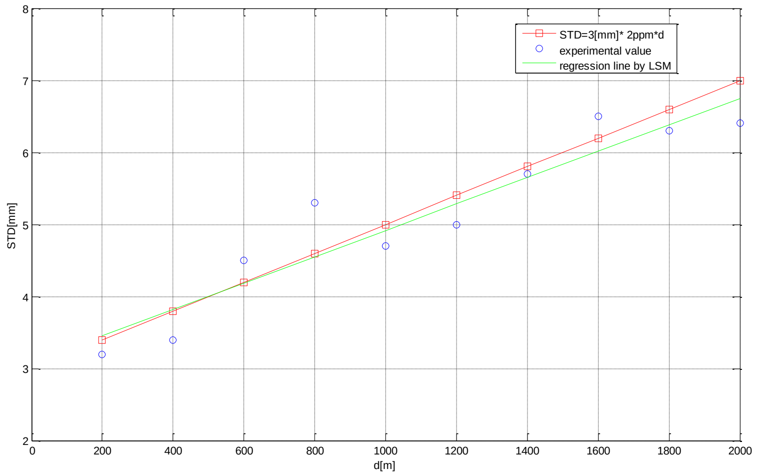

Fig. 6 Graph of theoretical values of the regression line adjusted by means of LSM

Tab. 1 Adjustment results of the regression line weighted with a normal distribution by means of LSM

no. of m.	Theoretical line			LSM		
	d	$\sigma_{d_{determ.}}$	$\sigma_{d_{exp.}}$	ε	v	Δ
	[m]	[mm]	[mm]	[mm]	[mm]	[mm]
1.	200.00	3.20	3.40	0.20	0.2473	-0.05
2.	400.00	3.40	3.80	0.40	0.4145	-0.01
3.	600.00	4.50	4.20	-0.30	-0.3182	0.02
4.	800.00	5.30	4.60	-0.70	-0.7509	0.05
5.	1000.00	4.70	5.00	0.30	0.2164	0.08
6.	1200.00	5.00	5.40	0.40	0.2836	0.12
7.	1400.00	5.70	5.80	0.10	-0.0491	0.15
8.	1600.00	6.50	6.20	-0.30	-0.4818	0.18
9.	1800.00	6.30	6.60	0.30	0.0855	0.21
10.	2000.00	6.40	7.00	0.60	0.3527	0.25

Regression line parameters:

Deterministic shape of the line   y = 3,0 [mm] + 2,0*ppm*d

Parameters of the line estimated by LSM   y = 3,1 [mm] + 1,8*ppm*d

Note:

$$STD = \sigma_d = b + a.d.ppm = 2d.ppm + 3$$

$$\Delta = \varepsilon - v\,,\ \ \varepsilon = \sigma_{d_{determ.}} - \sigma_{d_{exp.}}$$

Given the fact that the present contribution is devoted to the issues of estimation methods, which allow to reveal the so-called outlier measurements in the set of measured data and for which we do not select the arithmetic mean as the centrality parameter, the regression line is deliberately weighted with just one outlier, in order to track the performance of such methods, Fig. 6. Such modified regression line was adjusted first by the method of least squares and then by the iteration-based robust M-estimation methods.

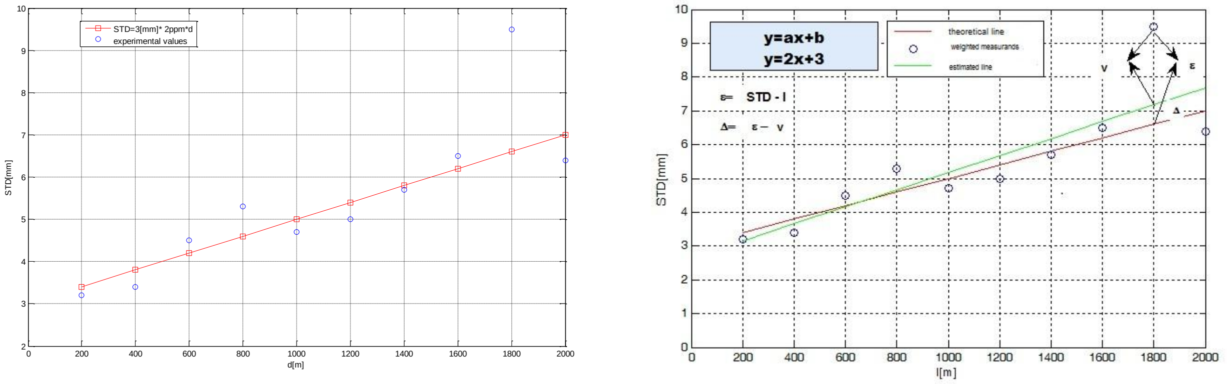

Fig. 7 Graph of theoretical values of the proposed regression line with an experimental outlier

The graphic interpretation of the results obtained by the adjustment of the modified regression line processed by LSM is shown in the following figure:

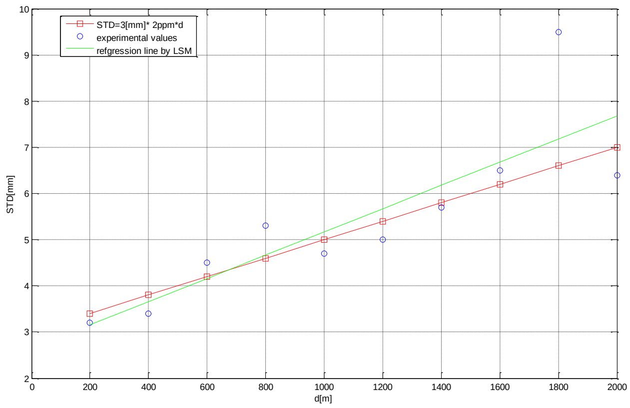

Fig. 8 Graphical interpretation of the adjustment results of the regression line with an experimental outlier by LSM

The numerical results of the line adjustment by the least squares method is presented in Tab. 2.

Tab. 2 Adjustment results of the regression line with an experimental outlier by LSM

no. of m.	Theoretical line			LSM		
	d	$\sigma_{d\,determ.}$	$\sigma_{d\,exp.}$	**ε**	**v**	**Δ**
	[m]	**[mm]**	**[mm]**	**[mm]**	**[mm]**	**[mm]**
1.	200.00	3.20	3.40	0.20	-0.0436	0.24
2.	400.00	3.40	3.80	0.40	0.2594	0.14
3.	600.00	4.50	4.20	-0.30	-0.3376	0.04
4.	800.00	5.30	4.60	-0.70	-0.6345	-0.07
5.	1000.00	4.70	5.00	0.30	0.4685	-0.17
6.	1200.00	5.00	5.40	0.40	0.6715	-0.27
7.	1400.00	5.70	5.80	0.10	0.4745	-0.37
8.	1600.00	6.50	6.20	-0.30	0.1776	-0.48
9.	1800.00	9.50	6.60	-2.90	-2.3194	-0.58
10.	2000.00	6.40	7.00	0.60	1.2836	-0.68

Regression line parameters:

Deterministic shape of the line y = 3.0 [mm] + 2.0*ppm*d

Parameters of the line estimated by LSM y = 2.7 [mm] + 2.5*ppm*d

4.2 Regression line adjustment by means of robust M-estimation methods

The robust M-estimator methods along with the Denmark method are based on the principle of the least squares method. These are iterative methods (in which the adjustment process is repeated several times); their principle consists in so-called *"reweighting"*, which means that the weights are being changed intentionally. In the first iteration step, the standard method of least squares is performed, where the weight of each measurement is the same and equal to one. In the next step, the weights are correction functions. First the regression line weighted with a normal distribution and then the regression line weighted with an outlier was adjusted through the above robust M-estimation methods and the Danish method.

The adjustment results of both lines are shown in Tables 4 and 5, together with the graphical interpretation of processing results.

Tab. 3 Comparison of the adjustment results of the regression line weighted with a normal distribution by means of LSM and robust M-estimation methods

	LSM					DANISH			HUBER			HAMPEL		
d	σ_d *exper.*	σ_d*deter*	ε	v	Δ	v	p	Δ	v	p	Δ	v	p	Δ
[m]	[mm]	[mm]		[mm]	[mm]	[mm]		[mm]	[mm]		[mm]	[mm]		[mm]
200.00	3.20	3.40	0 .20	0.2473	-0.05	0.2473	1	-0.05	0.2473	1	-0.05	0.2473	1	-0.05
400.00	3.40	3.80	0.40	0.4145	-0.01	0.4145	1	-0.01	0.4145	1	-0.01	0.4145	1	-0.01
600.00	4.50	4.20	-0.30	-0.3182	0.02	-0.3182	1	0.02	-0.3182	1	0.02	-0.3182	1	0.02
800.00	5.30	4.60	-0.70	-0.7509	0.05	-0.7509	1	0.05	-0.7509	1	0.05	-0.7509	1	0.05
1000.00	4.70	5.00	0.30	0.2164	0.08	0.2164	1	0.08	0.2164	1	0.08	0.2164	1	0.08
1200.00	5.00	5.40	0.40	0.2836	0.12	0.2836	1	0.12	0.2836	1	0.12	0.2836	1	0.12
1400.00	5.70	5.80	0.10	-0.0491	0.15	-0.0491	1	0.15	-0.0491	1	0.15	-0.0491	1	0.15
1600.00	6.50	6.20	-0.30	-0.4818	0.18	-0.4818	1	0.18	-0.4818	1	0.18	-0.4818	1	0.18
1800.00	6.30	6.60	0.30	0.0855	0.21	0.0855	1	0.21	0.0855	1	0.21	0.0855	1	0.21
2000.00	6.40	7.00	0.60	0.3527	0.25	0.3527	1	0.25	0.3527	1	0.25	0.3527	1	0.25

Regression line parameters:

Deterministic shape of the line — y = 3.0 [mm]+2,0*ppm*d
Parameters of the line estimated by LSM — y = 3.1 [mm]+1,8*ppm*d
Parameters of the line estimated by the Danish method — y = 3.1 [mm]+1,8*ppm*d
Parameters of the line estimated by the Huber method — y = 3.1 [mm]+1,8*ppm*d
Parameters of the line estimated by the Hampel method — y = 3.1 [mm]+1,8*ppm*d

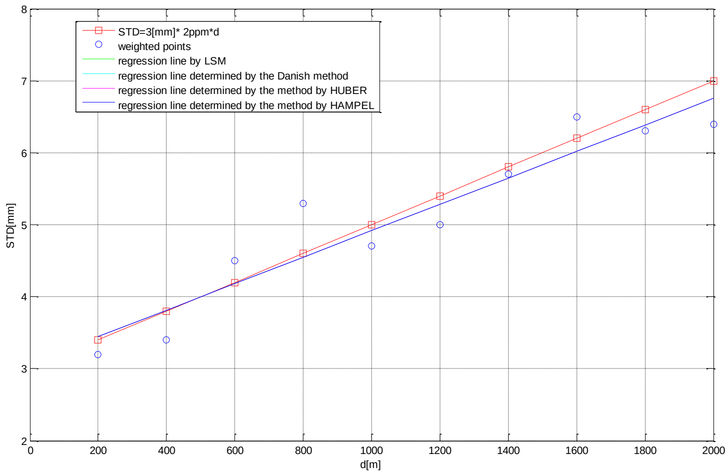

Fig. 9 Graphical interpretation of the adjustment results of the regression line weighted with a normal distribution by means of LSM and robust M-estimation methods

In case of the regression line non-weighted with any outlier the results of the adjustment by LSM and the robust M-estimation methods are identical as demonstrated in Tab. 4 and in the graphical interpretation of the processing results, Fig. 9. The comparison of the adjustment results of the modified regression line (weighted with an outlier) is presented in Tab. 4 and Fig. 10.

Tab. 4 Comparison of the adjustment results of the regression line weighted with a normal distribution by means of LSM and robust M - estimation methods

	LSM					DANISH			HUBER			HAMPEL		
d	σ_d *exper.*	σ_d*deter*	**ε**	**v**	**Δ**	**v**	**p**	**Δ**	**v**	**p**	**Δ**	**v**	**p**	**Δ**
[m]	[mm]	[mm]	[mm]	[mm]	[mm]	[mm]		[mm]	[mm]		[mm]	[mm]		[mm]
200.00	3.20	3.40	0.20	-0.0436	0.24	0.2369	1.0000	-0.04	0.2369	1.0000	-0.04	0.2365	1.0000	0.0000
400.00	3.40	3.80	0.40	0.2594	0.14	0.4090	1.0000	-0.01	0.4090	1.0000	-0.01	0.4088	1.0000	0.2167
600.00	4.50	4.20	-0.30	-0.3376	0.04	-0.3189	1.0000	0.02	-0.3189	1.0000	0.02	-0.3189	1.0000	-0.4667
800.00	5.30	4.60	-0.70	-0.6345	-0.07	-0.7468	1.0000	0.05	-0.7468	1.0000	0.05	-0.7466	1.0000	-0.8500
1000.00	4.70	5.00	0.30	0.4685	-0.17	0.2253	1.0000	0.07	0.2253	1.0000	0.07	0.2257	1.0000	0.1667
1200.00	5.00	5.40	0.40	0.6715	-0.27	0.2974	1.0000	0.10	0.2974	1.0000	0.10	0.2980	1.0000	0.2833
1400.00	5.70	5.80	0.10	0.4745	-0.37	-0.0305	1.0000	0.13	-0.0305	1.0000	0.13	-0.0298	1.0000	0.0000
1600.00	6.50	6.20	-0.30	0.1776	-0.48	-0.4584	1.0000	0.16	-0.4584	1.0000	0.16	-0.4575	1.0000	-0.3833
1800.00	9.50	6.60	-2.90	-2.3194	-0.58	-3.0863	0.1059	0.19	-3.0863	0.4860	0.19	-3.0852	0.6483	-2.9667
2000.00	6.40	7.00	0.60	1.2836	-0.68	0.3858	1.0000	0.21	0.3858	1.0000	0.21	0.3871	1.0000	0.5500

Regression line parameters:

Deterministic shape of the line — y = 3.0[mm]+2,0*ppm*d
Parameters of the line estimated by LSM — y = 2.7[mm]+2,5*ppm*d

Parameters of the line estimated by the Danish method	y = 3.1[mm]+1,9*ppm*d
Parameters of the line estimated by the Huber method	y = 3.1[mm]+1,9*ppm*d
Parameters of the line estimated by the Hampel method	y = 3.1[mm]+1,9*ppm*d

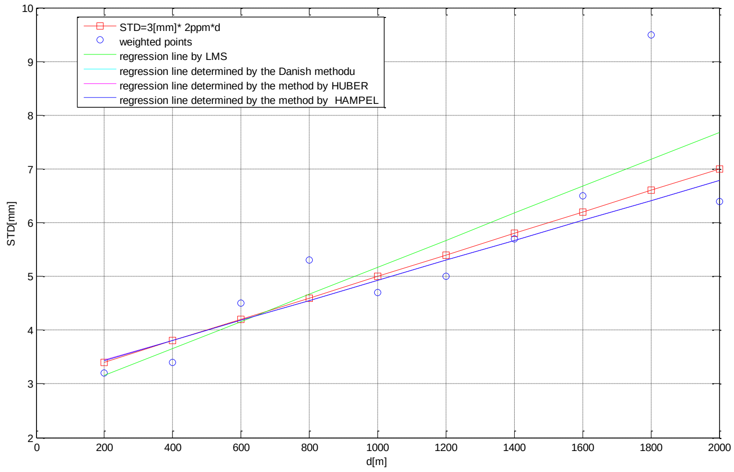

Fig. 10 Graphical interpretation of LSM and the robust M-estimation methods on the example of a regression line weighted with an experimental outlier

It is clear from the processing results that in this case the robust M-estimation methods came to identical results, herewith they assigned the largest correction and the smallest weight value to the outlier (Table 4); the graphical representations of the adjustment results of such modified line by means of the robust M-estimation methods are identical.

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RESUMÉ

V predloženom príspevku bol zámerne zvolený jednoduchý, ale o to názornejší príklad demonštrujúci pozitívne vlastnosti alternatívnych, v spracovaní geodetických meraní čoraz častejšie používaných a odporúčaných robustných M - odhadovacích metód na báze iteratívneho vyrovnania metódy najmenších štvorcov s účelovo znižovaným, deformujúcim vplyvom týchto chýb na odhadované parametre a pre ich striktnejšiu identifikáciu v súbore meraných veličín. Matematická báza týchto metód sa pomerne nenáročne implementuje do algoritmu metódy najmenších štvorcov a v práci uvedené M–robustné odhady podľa Hubera, Hampela alebo dánskej metódy poukazujú na vzájomnú tesnosť ich výsledkov.

Cieľom príspevku bolo poukázať na skutočnosť, že robustné odhadovanie metódy predstavujú silný nástroj na identifikáciu odľahlých meraní, ktoré z určitých objektívnych dôvodov prenikli do súborov dát meraných terénnych veličín. Záleží individuálne na každom prípade či odľahlé merania budú zo súborov meraných veličín eliminované, resp. či budú nahradené novými, nezávislými meraniami alebo analýzou ich genézy bude problém hlbšie skúmaný pre podrobnejšie a objektívnejšie popísanie experimentálneho merania a dynamiky stavu obklopujúceho fyzikálneho prostredia komplexnejšími matematicko–štatistickými modelmi.

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